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Abstract. A technique is developed which allows a unified treatment of *null*, *spacelike* and *timelike* infinities. Null and spacelike infinities have previously been treated using such techniques. The technique is here applied to the projective structure of timelike infinity and a number of examples worked out.

1. Introduction

Construction of a boundary with a well behaved metric on it has been achieved by Penrose (1968) for asymptotically flat spacetimes in *null* directions, by defining a suitable conformal metric. The conformal metric is then used to study the behaviour of zero rest mass fields at null infinity. Geroch (1972) has used a conformal completion by a single point of smooth three-dimensional spacelike hypersurfaces in a spacetime to define asymptotic flatness in *spatial* directions, and to study the behaviour of the gravitational field at spatial infinity.

Recently, Eardly (1970) and Eardly and Sachs (1973), by making a projective change of connection, have constructed boundaries for many spacetimes for past and future *timelike* infinities. The projective connection is defined in each case so as to remain well behaved at timelike infinity. The boundary in the projective space is defined as a set of points each of which is an equivalence class of timelike geodesics parallel in the infinite past or future.

In this paper the idea of a generalized conformal transformation is introduced. Roughly speaking, this is like a conformal transformation with different factors in different directions. More precisely, spacetime is foliated by a one-parameter family of smooth hypersurfaces and different conformal transformations are carried out (i) in the hypersurfaces of the family and (ii) in the space of normals. It will be shown that the Eardly–Sachs construction can be achieved using such a transformation and that the Penrose and Geroch constructions are (as is fairly obvious) also special cases.

2. Generalized conformal transformations

Let $(M, g_{\alpha\beta})$ be a smooth $(C^k, k \geqslant 2)$ connected time-orientable spacetime. Greek indices run over (1, 2, 3, 4) and Latin indices run over (1, 2, 3).

Consider a set of three-dimensional spacelike hypersurfaces S in M and let η_α be the unit normal vector field to the hypersurfaces. The metric for the space of normals to the hypersurfaces is

$$\eta_{\alpha\beta} = \eta_\alpha \eta_\beta. \tag{1}$$

The indices of $\eta_{\alpha\beta}$ are raised and lowered with $g_{\alpha\beta}$.

Write

$$g_{\alpha\beta} = \eta_{\alpha\beta} + (g_{\alpha\beta} - \eta_{\alpha\beta}). \tag{2}$$

Then $(g_{\alpha\beta} - \eta_{\alpha\beta})$ is the metric for the set of spacelike hypersurfaces S. A *generalized conformal transformation* is a change of metric,

$$g_{\alpha\beta} \to \hat{g}_{\alpha\beta} = e^{2\psi}\eta_{\alpha\beta} + e^{2\Omega}(g_{\alpha\beta} - \eta_{\alpha\beta}), \tag{3}$$

where ψ and Ω are smooth functions on M.

Three relations of interest between ψ and Ω arise.

(i) $\psi = \Omega \neq 0$. This is a conformal transformation, used for treating *null* infinity.

(ii) $\psi = 0, \Omega \neq 0$. In this case, a conformal transformation is applied in each spacelike hypersurface only. This is the transformation used in Geroch's treatment of *spatial* infinity.

(iii) $\psi \neq 0, \Omega \neq 0$ and $\psi \neq \Omega$. This will be called a *quasi-conformal transformation.* It will be shown that, with a suitable choice of ψ and Ω the transformation induces a projective change of connection with which the Eardly–Sachs boundaries may be reconstructed.

For any of the above cases, the inverse of $\hat{g}_{\alpha\beta}$ is

$$\hat{g}^{\alpha\beta} = e^{-2\psi}\eta^{\alpha\beta} + e^{-2\Omega}(g^{\alpha\beta} - \eta^{\alpha\beta}). \tag{4}$$

The induced change of connection is

$$\Gamma^\lambda_{\alpha\beta} \to \hat{\Gamma}^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} + \delta^\lambda_\alpha \Omega_{,\beta} + \delta^\lambda_\beta \Omega_{,\alpha} + \Lambda^\lambda_{\alpha\beta}, \tag{5}$$

where the comma denotes a partial derivative and $\Lambda^\lambda_{\alpha\beta} = \Lambda^\lambda_{\beta\alpha}$ is a complicated expression given explicitly in the appendix.

Any projective change of connection has the form

$$\Gamma'^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} + \delta^\lambda_\alpha q_\beta + \delta^\lambda_\beta q_\alpha, \tag{6}$$

where q_α is a covariant vector field. Geodesics remain invariant under such transformations.

Comparison of (5) and (6) suggests one obvious way to construct a generalized conformal transformation which is also projective, namely to set

$$\Lambda^\lambda_{\alpha\beta} = 0. \tag{7}$$

Then (5) is a projective change of connection with

$$\Omega_{,\alpha} = q_\alpha. \tag{8}$$

Now work in a coordinate patch in which (cf equation (2))

$$\begin{aligned} g_{i4} &= 0 \\ g_{44} &= \eta_{44} \\ \eta_{ij} &= \eta_{i4} = 0. \end{aligned} \tag{9}$$

Equations (7) then reduce to

$$\psi_{,4} = 2\Omega_{,4} \tag{10a}$$

$$g_{ij}\Omega_{,4} + \tfrac{1}{2}(1 - e^{2\psi - 2\Omega})g_{ij,4} = 0 \tag{10b}$$

$$\psi_{,i} = \Omega_{,i} \tag{10c}$$

$$\left(g_{44} - \frac{\eta_{44}}{e^{2\psi - 2\Omega} - 1}\right)\Omega_{,j} + \frac{\eta_{44}}{1 - e^{2\Omega - 2\psi}}\psi_{,j} + \tfrac{1}{2}(e^{2\psi - 2\Omega} - 1)\eta_{44,j} = 0 \tag{10d}$$

$$g^{il}g_{jk}\Omega_{,l} = 0. \tag{10e}$$

3. The Eardly–Sachs asymptotic conditions

Eardly (1970) and Eardly and Sachs (1973) introduce a coordinate patch in which x^4 is a timelike function w, decreasing into the future and zero in the infinite future. Functions ψ and Ω will be found such that (i) the metric and the connection remain well behaved throughout the coordinate patch and (ii) the change in the connection is a projective one, ie, equations (10*a*)–(10*e*) are satisfied. Then it can be shown (Eardly 1970, Eardly and Sachs 1973) that a point on the surface W is an equivalence class of timelike geodesics parallel in the infinite future.

Let a coordinate patch P be chosen in M with coordinates (Eardly 1970) $\{x^\alpha\} = \{w, x^i\} \in O_\alpha$ where $O_\alpha = (0, w_0) \times O_i$, and O_i is an open subset of $\mathbb{R}_3$. Let W denote the surface $w = 0$. Now define $\bar{P} = P \cup W \subset M \cup W$; the coordinates on $\bar{P}$ are $\{\bar{x}^\alpha\} \in [0, w_0) \times O_i = \bar{O}_\alpha$. Consider $C^k (k \geqslant 2)$ manifolds admitting in such a coordinate patch the metric form

$$ds^2 = w^{-2n} e^{2F} dw^2 - w^{-n} L_{ij}\, dx^i\, dx^j, \tag{11}$$

where

(*a*) $F(w)$ and $L_{ij}(x^\alpha)$ are C^{k-1} on $\bar{O}_\alpha$,
(*b*) $F = 0$ for $w = 0$,
(*c*) L_{ij} is a positive definite metric on O_i for each $w \in [0, w_0)$,
(*d*) $n \geqslant 1$ is an integer.

The singular terms on W in the components of the Christoffel symbols for such a metric are

$$\Gamma^4_{44} = -\frac{n}{w} + F_{,4} \tag{12a}$$

$$\Gamma^i_{4j} = -\delta^i_j \frac{n}{2w} + \frac{1}{2} L^{ik} L_{kj,4}. \tag{12b}$$

With the coordinate system chosen as above, set

$$\psi_{,i} = \Omega_{,i} = 0.$$

Then (10*c*) to (10*e*) are satisfied and the only equations remaining to be solved are (10*a*) and (10*b*). Equation (10*a*) gives

$$\psi = 2\Omega + C \tag{13}$$

where C is a constant of integration.

Having obtained this, we can now check that the necessary conditions (10*a*) to (10*e*) are also sufficient for the quasi-conformal transformation to correspond to a projective change of the connection. A necessary and sufficient condition that two metrics for a spacetime be projectively related is that (Schouten 1954)

$$10q_\alpha = \partial_\alpha \ln \frac{\hat{g}}{g}, \tag{14}$$

where q_α is the covariant vector field in (6) and $\hat{g}$ and g are the determinants of the two metrics.

From (13) and (3) on the other hand it follows that

$$\frac{\hat{g}}{g} = e^{10\Omega}.$$

Hence

$$q_\alpha = \Omega_{,\alpha},$$

which is consistent with the assumption (8). The problem now reduces to that of solving (10*b*) and Ω so obtained must eliminate the singular terms in (12*a*) and (12*b*).

4. Examples

4.1. Minkowski spacetime

The metric can be written in the form

$$ds^2 = w^{-4}\, dw^2 - w^{-2}[d\chi^2 + \sinh^2\chi(d^2\theta + \sin^2\theta\, d\phi^2)],$$

where

$$t = \frac{1}{w}\cosh\chi$$

$$r = \frac{1}{w}\sinh\chi$$

and $(t, r, \theta, \phi) \in (-\infty, \infty) \times [0, \infty) \times [0, \pi] \times [0, 2\pi)$ are the usual polar coordinates for the Minkowski spacetime.

Equation (10*b*) gives

$$\Omega = \ln w - \tfrac{1}{2}\ln(1 + w^2),$$

$$e^{2\Omega} = \frac{w^2}{1 + w^2}.$$

(The constants of integration obtained when solving (10*a*) and (10*b*) can result only in scaling factors on the metric on *W* and are chosen here, as in all other examples, to make the scaling factor unity.)

Hence

$$\Omega_{,4} = \frac{1}{w} - \frac{w}{1 + w^2} = q_4$$

$$\Omega_{,i} = 0 = q_i.$$

With q_α given as above, the singular terms in (12*a*) and (12*b*) are eliminated. The quasi-conformally related metric is

$$d\hat{s}^2 = \frac{1}{(1+w^2)^2}\,dw^2 - \frac{1}{1+w^2}[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)].$$

Putting $w = 0$, the metric on the spacelike boundary is

$$dl^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta\,d\phi^2)$$

The surface is a three-pseudosphere.

(To agree with the notation of Eardly and Sachs (1973) write

$$dl^2 = d\chi^2 + \sinh^2\chi\,d\Psi_2^2,$$

where

$$d\Psi_2^2 = d\theta^2 + \sin^2\theta\,d\phi^2$$

and in general

$$d\Psi_n^2 = d\alpha^2 + \sin^2\alpha\,d\Psi_{n-1}^2,$$

where α is a coordinate.)

4.2. *De Sitter spacetime*

The metric can be written (Heckman and Schücking 1962)

$$ds^2 = dt^2 - \cosh^2 t\,d\Psi_3^2,$$

where $-\infty < t < \infty$. Substitution of $w = e^{-2t}$ yields

$$ds^2 = w^{-2}\,dw^2 - w^{-1}(1+w)^2\,d\Psi_3^2.$$

Using this, equation (10*b*) gives

$$\Omega = \tfrac{1}{2}\ln w - \tfrac{1}{2}\ln(w^2 + 3w + 1)$$

$$\Omega_{,4} = \frac{1}{2w} - \frac{w + \frac{3}{2}}{w^2 + 3w + 1} = q_4.$$

Also

$$\Omega_{,i} = 0 = q_i.$$

The singular terms in (12*a*) and (12*b*) are again eliminated. The quasi-conformal metric is

$$d\hat{s}^2 = \frac{dw^2}{(w^2 + 3w + 1)^2} - \frac{(1+w)^2}{(w^2 + 3w + 1)}\,d\Psi_3^2.$$

Putting $w = 0$, the metric on W is $d\Psi_3^2$.

4.3. *Friedman model with $k = 0$*

The metric is (Heckman and Schücking 1962)

$$ds^2 = dt^2 - t^{4/3}\,dE_3^2,$$

where $t \in (0, \infty)$ and dE_3^2 is the metric of a flat three-space.

Let

$$t = \tfrac{1}{3}w^{-3}.$$

Then

$$\mathrm{d}s^2 = w^{-8}\,\mathrm{d}w^2 - w^{-4}\,\mathrm{d}E_3^2,$$

and (10*b*) gives

$$\Omega = 2\ln w - \tfrac{1}{2}\ln(1+w^4)$$

$$\Omega_{,4} = \frac{2}{w} - \frac{2w^3}{1+w^4} = q_4$$

$$\Omega_{,i} = 0 = q_i.$$

The connection becomes nonsingular again and the quasi-conformal metric is

$$\mathrm{d}\hat{s}^2 = \frac{1}{(1+w^4)^2}\,\mathrm{d}w^2 - \frac{1}{1+w^4}\,\mathrm{d}E_3^2.$$

The surface W is flat.

4.4. *Friedman model with $k = -1$*

The metric can be written in the form (Heckman and Schücking 1962)

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\,\mathrm{d}l^2,$$

where $\mathrm{d}l^2$ is the line element of a three-pseudosphere and $t = a(\sinh\eta - \eta)$, $R(t) = a(\cosh\eta - 1)$ with a as a constant and η a parameter, $\eta \in (0, \infty)$.

Choose (Eardly 1970)

$$w = \frac{2}{a(\exp\eta - 1)}$$

and

$$a = 2.$$

Then

$$\mathrm{d}s^2 = w^{-4}(1+w)^{-4}\,\mathrm{d}w^2 - w^{-2}(1+w)^{-2}\,\mathrm{d}l^2.$$

(10*b*) now gives

$$\Omega = \ln w + \ln(1+w) - \tfrac{1}{2}\ln[w^2(1+w)^2+1]$$

$$\Omega_{,4} = \frac{1}{w} + \frac{1}{1+w} - \frac{2w^3+3w^2+w}{w^2(1+w)^2+1} = q_4.$$

Further,

$$\Omega_{,i} = 0 = q_i.$$

The quasi-conformal metric is again well behaved and the transformed metric is

$$\mathrm{d}\hat{s}^2 = \frac{1}{[w^2(1+w)^2+1]^2}\,\mathrm{d}w^2 - \frac{1}{w^2(1+w)^2+1}\,\mathrm{d}l^2.$$

The metric on W is $\mathrm{d}l^2$. W is a three-pseudosphere.

4.5. *The Schwarzschild spacetime*

The usual metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - 2M/r} - r^2 \, d\Psi_2^2$$

can be transformed, using the substitution $t = w^{-1}\cosh\alpha$, $r = w^{-1}\sinh\alpha$ and $\sinh\alpha = \sinh\chi + wh(w, \sinh\chi)$ where h is regular, into the form (Eardly 1970)

$$ds^2 = w^{-4}\, dw^2\left(1 - 2Mw\frac{\cosh^2\chi + \sinh^2\chi}{\sinh\chi} + O(w^2)\right)$$
$$- w^{-2}[(1 + O(w))\, d\chi^2 + \sinh^2\chi(1 + O(w))\, d\Psi_2^2]. \qquad (16)$$

The range of χ is chosen $\epsilon < \chi < \infty$, where $\epsilon > 0$, to avoid the black hole.

Since the function F in (11) is here $f(w, x^i)$, it is evident that $\eta_{44,j} \neq 0$ and we cannot identically satisfy (10*c*) to (10*e*) by setting $\Omega_{,j} = \psi_{,j} = 0$. Write (16) in the form

$$ds^2 = w^{-4}\, e^{2f}\, dw^2 - w^{-2}[(1 + O(w))\, d\chi^2 + \sinh^2\chi(1 + O(w))\, d\Psi_2^2]. \qquad (17)$$

Consider the conformally related metric

$$d\tilde{s}^2 = e^{-2f}\, ds^2. \qquad (18)$$

Then (10*c*) to (10*e*) are satisfied using $d\tilde{s}^2$ and we are left with (10*b*). The exact form of the solution of (10*b*) away from W need not concern us here. In the limit

$$w \to 0,$$

(18) tends to

$$d\tilde{s}^2 = w^{-4}\, dw^2 - w^{-2}(d\chi^2 + \sinh^2\chi\, d\chi_2^2).$$

The solution for Ω in (10*b*) is as in the flat case with

$$\Omega = \ln w - \tfrac{1}{2}\ln(1 + w^2)$$

and

$$d\hat{s}^2 \simeq dw^2 - (d\chi^2 + \sinh^2\chi\, d\Psi_2^2).$$

Let

$$d\hat{s}^2 \simeq e^{2\sigma} w^{-4}\, dw^2 - e^{2\gamma} w^{-2}[(1 + O(w))\, d\chi^2 + (1 + O(w))\sinh^2\psi\, d\Psi_2^2]$$

where

$$e^{2\sigma} = w^4\, e^{-2f}, \qquad e^{2\gamma} = w^2\, e^{-2f}.$$

It is evident that the quasi-conformal factors σ and γ now represent a projective change of connection plus a conformal transformation. On W, where the approximations become exact, σ and γ will represent a projective change of connection only. W is a three-pseudosphere as in the Minkowski case but the range of χ is chosen to avoid the black hole.

5. Conclusion

It seems that the success of the 'generalized conformal transformation' technique in treating infinity in the three directions is related intimately to that of choosing a suitable coordinate system. The metric expressed in these coordinates is conformally or quasi-conformally related to one which is well behaved at infinity in any of the three directions under consideration. The conformal or quasi-conformal factors diverge at infinity and are removed by a generalized conformal transformation.

The impossibility of any smooth connection between an Eardly–Sachs *timelike* boundary and a Penrose *null* boundary is shown by equation (13). This is why, topologically, gluing of these two boundaries is non-Hausdorff (R K Sachs, private communication).

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Appendix

The expression for $\Lambda^{\lambda}_{\alpha\beta}$ can most easily be obtained by writing (3) in the form

$$\hat{g}_{\alpha\beta} = e^{2\Omega} g_{\alpha\beta} \mp e^{2\gamma} \eta_{\alpha\beta} \tag{A.1}$$

where

$$e^{2\gamma} = e^{2\psi} - e^{2\Omega}$$

and

$$\hat{g}^{\alpha\beta} = e^{-2\Omega} g^{\alpha\beta} + e^{-2\sigma} \eta^{\alpha\beta} \tag{A.2}$$

where

$$e^{-2\sigma} = e^{-2\psi} - e^{-2\Omega}.$$

Then

$$\begin{aligned}\Lambda^{\lambda}_{\alpha\beta} = & -g^{\lambda\nu} g_{\alpha\beta} \Omega_{,\nu} + e^{2\gamma - 2\Omega} \{ g^{\lambda\nu} [\alpha\beta, \nu]_{\eta} + \eta^{\lambda}_{\alpha} \gamma_{,\beta} + \eta^{\lambda}_{\beta} \gamma_{,\alpha} - g^{\lambda\nu} \eta_{\alpha\beta} \gamma_{,\nu} \} \\ & + e^{2\Omega - 2\sigma} \{ \eta^{\alpha\beta} [\alpha\beta, \nu]_{g} + \eta^{\lambda}_{\alpha} \Omega_{,\beta} + \eta^{\lambda}_{\beta} \Omega_{,\alpha} - \eta^{\lambda\nu} g_{\alpha\beta} \Omega_{,\nu} \} \\ & + e^{2\gamma - 2\sigma} \{ \eta^{\lambda\nu} [\alpha\beta, \nu]_{\eta} + \eta^{\lambda}_{\alpha} \gamma_{,\alpha} - \eta^{\lambda\nu} \eta_{\alpha\beta} \gamma_{,\nu} \}\end{aligned}$$

where

$$[\alpha\beta, \nu]_{\eta} = \tfrac{1}{2} (\eta_{\alpha\nu,\beta} + \eta_{\beta\nu,\alpha} - \eta_{\alpha\beta,\nu})$$

$$[\alpha\beta, \nu]_{g} = \tfrac{1}{2} (g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}).$$

References

Eardly D 1970 *PhD Thesis* University of California
Eardly D and Sachs R K 1973 *J. math. Phys.* **14** 209
Geroch R 1972 *J. math. Phys.* **13** 956–68
Heckman E and Schücking O 1962 *Gravitation: An Introduction to Current Research* ed L Witten (New York: Wiley)
Penrose R 1968 *Battelle Rencontres 197 Lectures in Mathematics and Physics* ed C M DeWitt and J A Wheeler (New York: Benjamin)
Schouten J A 1954 *Ricci Calculus* (Berlin: Springer-Verlag)